# Euler and the Multiplication Formula for the $\Gamma$-Function.

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We show that the multiplication formula for the $\Gamma$-function was already found by Euler in [$E$421], although it is usually attributed to Gauss.

## Contents

# 1 INTRODUCTION

In modern notation, the multiplication formula for the Γ-function reads as follows:

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{x-\frac{1}{2}}}\cdot\Gamma(x). \tag{1}$$

Here, $n$ is a natural number and

$$\Gamma(x) := \int\limits_0^\infty e^{-t}t^{x-1}dt \quad \text{for} \quad Re(x) > 0.$$

(1) is often called the Gaussian multiplication formula and it was indeed also proven by Gauss in [*Ga*28]. But we will show, how it can be derived from the results given by Euler in his lesser-known paper [*E*421], where he gave a formula equivalent to it.

# 2 PREPARATIONS

Of course, we need some formulas given by Euler in [*E*421] and some auxiliary formulas that can easily derived from Euler's formulas.

## 2.1 EXPRESSING EULER'S FORMULA IN MODERN NOTATION

In § 44 of his paper [*E*421] [1] Euler defines

$$\left(\frac{p}{q}\right) = \int\limits_0^1 \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}}.$$

By the substitution $x^n = y$ it is easily seen that

$$\left(\frac{p}{q}\right) = \frac{1}{n}\int\limits_0^1 y^{\frac{p}{n}-1}dy(1-y)^{\frac{q}{n}-1} = \frac{1}{n}B\left(\frac{p}{n},\frac{q}{n}\right) \tag{2}$$

where

[1] See page 343 in the Opera Omnia Version. He also studies these formulas extensively in [*E*321].

$$B(x,y) = \int\limits_0^1 t^{x-1}dt(1-t)^{y-1} \quad \text{for} \quad \mathrm{Re}(x), \mathrm{Re}(y) > 0$$

is the Beta function. Euler implicitly assumes $p$ and $q$ to be natural numbers in $\left(\frac{p}{q}\right)$[2].

## 2.2 Two auxiliary formulas

### 2.2.1 First Formula

We have

$$\prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right) = \pi^{\frac{n-1}{2}} \sqrt{\prod_{i=1}^{n-1} \sin\left(\frac{i\pi}{n}\right)}. \tag{3}$$

To see this, consider the reflection formula for the $\Gamma$-function, also given by Euler in § 43 of [$E$421][3],

$$\frac{\pi}{\sin \pi x} = \Gamma(x)\Gamma(1-x).$$

Now, just apply this for $x = \frac{i}{n}$ with $i = 1, 2, \cdots n-1$. Then

$$\left.\begin{array}{rcl} \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right) & = & \frac{\pi}{\sin\frac{\pi}{n}} \\ \Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{n-2}{n}\right) & = & \frac{\pi}{\sin\frac{2\pi}{n}} \\ \vdots & & \vdots \\ \Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{1}{n}\right) & = & \frac{\pi}{\sin\frac{(n-1)\pi}{n}} \end{array}\right\} \Rightarrow \prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right)^2 = \pi^{\frac{n-1}{2}} \prod_{i=1}^{n-1} \sin\left(\frac{i\pi}{n}\right)$$

(3) now follows by taking the square root.

[2]This restriction is not necessary, of course.

[3]In the Opera Omnia Version the formula can be found on page 342. He writes it as $[\lambda] \cdot [-\lambda] = \frac{\lambda\pi}{\sin \pi\lambda}$ and $[\lambda]$ stands for $\lambda!$

### 2.2.2 Second Formula

We have

$$\prod_{i=1}^{n-1} \sin\left(\frac{i\pi}{n}\right) = \frac{n}{2^{n-1}}. \tag{4}$$

This is an elementary formula. You can find a derivation, e.g., in [*Fr*06] (p. 17, ex. 19).

## 3 Derivation of the Product Formula

In § 53 of [*E*421][4] Euler gives the formula

$$\left[\frac{m}{n}\right] = \frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)}.$$

Euler uses $[x]$ to denote the factorial of $x$ so that in our notation $\left[\frac{m}{n}\right] = \Gamma\left(\frac{m}{n}+1\right)$. $m$ and $n$ are natural numbers, at least for Euler. But we see that we can interpolate $1\cdot 2\cdot 3\cdots(m-1)$ by $\Gamma(m)$. Therefore, let us assume $x$ to be real and $x > 0$. Further, using (2) Euler's formula reads

$$\Gamma\left(\frac{x}{n}\right) = \sqrt[n]{n^{n-x}\Gamma(x)\frac{1}{n^{n-1}}B\left(\frac{1}{n},\frac{x}{n}\right)B\left(\frac{2}{n},\frac{x}{n}\right)\cdots B\left(\frac{n-1}{n},\frac{x}{n}\right)}.$$

Now, we have the following relation among the $B$- and $\Gamma$- function, also given by Euler in the Supplement of [*E*421] [5].

$$B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}.$$

Substituting the right-hand side for each $B$-function and after some simplification under the $\sqrt[n]{}$-sign we will find

$$\Gamma\left(\frac{x}{n}\right) = \sqrt[n]{n^{1-x}\Gamma(x)\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+1}{n}\right)}\cdot\frac{\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+2}{n}\right)}\cdots\frac{\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+n-1}{n}\right)}}.$$

[4]See page 348 in the Opera Omnia Version.

[5]See page 354 in the Opera Omnia Version.

Now, let us simplify this by bringing all Γ-functions of fractional argument to the left-hand side. We will find

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\Gamma\left(\frac{x+2}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right)=n^{1-x}\Gamma(x)\Gamma\left(\frac{1}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right)$$

The product on the right-hand side, $\Gamma\left(\frac{1}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right)$, can be simplified by means of equation (3) and (4) so that we arrive at

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\Gamma\left(\frac{x+2}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right)=n^{1-x}\Gamma(x)\sqrt{\pi^{n-1}\frac{2^{n-1}}{n}}=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-x}\Gamma(x).$$

Thus, we arrived at the multiplication formula (1).

## 4 Conclusion

Now we saw that Euler already found the multiplication formula of the Γ-function. He just expressed it in terms of the symbol $\left(\frac{p}{q}\right)$ or, in modern notation, in terms of $B(p,q)$. He probably did not transform it, as we did here, since he wanted to express the factorial of rational numbers in terms of integrals of *algebraic* functions. Therefore, he probably also did not replace $1\cdot 2\cdot 3\cdots(m-1)$ by $\Gamma(m)$.

Therefore, this paper of Euler should provide some motivation to go through other papers written by Euler carefully and try to find some more results he discovered (although he might expressed it differently), but which are attributed to others. This will certainly be of interest for anyone studying the history of mathematics.

## References


[E321] L. Euler *Observationes circa integralia formularum* $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ *posito post integrationem* $x=1$, Melanges de philosophie et de la mathematique de la societe royale de Turin 3, 1766, pp. 156-177"', reprinted in "'*Opera Omnia*: Series 1, Volume 17, pp. 269 - 288

[E421] L. Euler *Evolutio formulae integralis* $\int x^{f-1}dx\,(\log(x))^{\frac{m}{n}}$ *integratione a valore* $x=0$ *ad* $x=1$ *extensa* , Novi Commentarii academiae scientiarum

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[$Fr$06] E. Freitag, R. Busam *Funktionentheorie 1*, Springer (4. korrigierte und erweiterte Ausgabe)

[$Ga$28] C. Gauss *Disquisitiones generales circa seriem infinitam* $1 + \frac{a \cdot b}{1 \cdot c}x + \frac{a(a+1) \cdot b(b+1)}{1 \cdot 2 \cdot c(c+1)}x^2 + \cdots$, *Commentationes recentiores Bd. II. , Göttingen 1813*